Quadric invariants and degeneration in smooth-étale cohomology

Saurav Bhaumik and Nitin Nitsure

Dedicated to M.S. Narasimhan on his completing 80 years

Abstract

For a regular pair (X, Y) of schemes over $\mathbb{Z}[1/2]$ of pure codimension 1, we consider quadric bundles on X which are nondegenerate on $X - Y$, but are minimally degenerate on Y. We give a formula for the behaviour of the cohomological invariants (characteristic classes) of the nondegenerate quadric bundle on $X - Y$ under the Gysin boundary map to the étale cohomology of Y with mod 2 coefficients.

The results here are the algebro-geometric analogs of topological results for complex bundles proved earlier by Holla and Nitsure, continuing further the algebraization program which was commenced with a recent paper by Bhaumik. We use algebraic stacks and their smooth-étale cohomologies, A^1-homotopies and Gabber's absolute purity theorem as algebraic replacements for the topological methods used earlier, such as CW complexes, real homotopies, Riemannian metrics and tubular neighbourhoods. Our results also hold for quadric bundles over algebraic stacks over $\mathbb{Z}[1/2]$.

2010 Math. Sub. Class. : 14F20, 14D06, 14D23.

Contents

1 Introduction

This paper extends to the algebraic category the topological results of Holla and Nitsure [H-N-1] and [H-N-2]. We address the following three questions, which were answered in the earlier papers in the topological category.

Let S be a scheme on which 2 is invertible, for example, $S = \mathrm{Spec}\,\mathbb{Z}[1/2]$. Let a nondegenerate quadratic triple $T = (E, L, b)$ of rank n on a scheme X over S consist of a rank n vector bundle E and a line bundle L on X, and a nondegenerate symmetric bilinear form b on E taking values in L. As we explain later, such a triple T is the same as a principal $GO_{n,S}$-bundle where $GO_{n,S}$ is the general orthogonal group-scheme over S. Let $BGO_{n,S}$ be its classifying algebraic stack, and let $H^*(BGO_{n,S}, \mathbb{F}_2)$ denote its cohomology ring in the smooth-étale topology. A characteristic class of degree i for nondegenerate quadratic triples of rank n is an element $\alpha \in H^i(BGO_{n,S}, \mathbb{F}_2)$. Such an α associates to any such T an étale cohomology class $\alpha(T) \in H^i(X, \mathbb{F}_2)$, which is functorial in the sense that it commutes with pull-backs under any $X' \to X$.

Question 1 What is the ring $H^*(BGO_{n,S}, \mathbb{F}_2)$ of characteristic classes for nondegenerate quadratic triples of rank n on schemes over S?

A nondegenerate quadric bundle Q on X of rank n will mean an equivalence class of rank n nondegenerate quadratic triples on X, where we regard a triple $T = (E, L, b)$ as being equivalent to another triple $T \otimes K = (E \otimes K, L \otimes K^2, b \otimes 1_K)$, where K is any line bundle on X (note that the vanishing of the quadratic forms associated to b and $b \otimes 1_K$ define the same closed subscheme $C \subset P(E) = P(E \otimes K)$, which a bundle of nondegenerate quadrics). A cohomological invariant of degree i associates to any such Q an étale cohomology class $\alpha(Q) \in H^i(X, \mathbb{F}_2)$, which is functorial under pull-backs. These form a subring $PH^*(BGO_{n,S}, \mathbb{F}_2) \subset H^*(BGO_{n,S}, \mathbb{F}_2)$.

Question 2 What is the ring $PH^*(BGO_{n,S}, \mathbb{F}_2)$ of cohomological invariants for nondegenerate quadric bundles of rank n on schemes over S?

We will now come to the third question, answering which is the main purpose of this paper. Let X be a regular scheme on which 2 is invertible, and let $Y \subset X$ be a closed regular subscheme of pure codimension 1. We are interested in quadric bundles over X which are nondegenerate over $X - Y$ and minimally degenerate over Y. This means that if Q is represented by a quadratic triple $T = (E, L, b)$, then b is nondegenerate of rank n on $X - Y$, while rank of $b|_Y$ is $n - 1$.

Question 3 How do the cohomological invariants of the nondegenerate quadric bundle $Q|_{X-Y}$ behave under degeneration?

Before we come to describing our solution to the question 3, we make some comments on questions 1 and 2. Let $QuasiProj_S$ be the category of all quasi-projective schemes over S. For any group scheme G over S, we have a functor $|BG|$ on $QuasiProj_S$, which associates to any X the set of all isomorphism classes of étale locally trivial G-torsors on X. Let $H^*(-, \mathbb{F}_2)$ be the functor on $QuasiProj_S$ which associates to any X the étale cohomology $H^*(X, \mathbb{F}_2)$. We show in section

4 (Proposition 10) that when the base S is an excellent Dedekind domain, for any reductive group-scheme G over S, any cohomology class $\alpha \in H^*(BG, \mathbb{F}_2)$ is the same as a morphism of functors α' from $|BG|$ to $H^*(-, \mathbb{F}_2)$ on the category $QuasiProj_S$. Bhaumik [Bh] has recently shown this when the base S is of the form $S = \operatorname{Spec} k$ where k is a field of characteristic $\neq 2$. All that remains for us to do is to point out how the arguments in [Bh] – which were based on an idea of Totaro [T] for approximating classifying stacks BG using Geometric Invariant Theory and Jouanolou's trick – extend from base k to base S, using the extension of GIT to more general bases due to Seshadri [Se].

Our second comment on questions (1) and (2) is as follows. The singular cohomology rings R_n of the CW complexes $BGO_n(\mathbb{C})$ for the complex Lie groups $GO_n(\mathbb{C})$ with $\mathbb{F}_2$ coefficients have been explicitly determined in Holla and Nitsure [H-N-1] and [H-N-2], in terms of generators and relations. The primitive subrings $PR_n \subset R_n$ have been determined in [H-N-2] in a closed form for all odd n, for $n = 4$, and for all n of the form $4m + 2$. As shown in [Bh], for any separably closed base field k of characteristic 2, we get the same answers: $H^*(BGO_{n,k}, \mathbb{F}_2) = R_n$ and $PH^*(BGO_{n,k}, \mathbb{F}_2) = PR_n$. The situation over a general base, such as $S = \operatorname{Spec} \mathbb{Z}[1/2]$, is more complicated (see Remark 25), and will not be described here.

The answer to the question (3) is the main theorem of this paper (Theorem 11), which describes how the étale cohomological invariants of the nondegenerate quadric bundle over $X - Y$ behave under degeneration. Concretely, we describe how the quadric invariants behave under the Gysin boundary map $\partial : H^*(X - Y, \mathbb{F}_2) \to H^{*-1}(Y, \mathbb{F}_2)$ from the étale cohomology of $X - Y$ to the étale cohomology of the degeneration locus Y.

An analogous result has been proved in [H-N-2] over complex numbers in the topological category using singular cohomology. Some of the crucial techniques used in the topological case – namely, tubular neighbourhoods, orthogonal projections using Riemannian metrics and usual homotopy – do not extend to the algebraic category. We replace the older arguments with the use of algebraic stacks which leads to a much more natural argument – one that also sheds light on the earlier topological result. A by-product of the proof is that the main theorem works also for degenerating quadric bundles defined over regular pairs of algebraic stacks over S, so we have stated and proved it in this generality.

Extensive concrete calculations for the rings R_n and PR_n have been given in [H-N-1] and [H-N-2], and the images of the generators of PR_n under the Gysin boundary map have been listed in all low ranks (up to rank 6) in [H-N-2]. As explained in section 5.5, these calculations now extend to the algebraic category over separably closed fields of characteristic $\neq 2$. We expect this to be of some interest, given the important role that quadratic forms play in arithmetic and algebraic geometry.

This paper is arranged as follows. In section 2, we recall basic facts about absolute purity over algebraic stacks, and prove a 'multiplicity lemma' (Lemma 5) for Gysin boundary maps which is useful later. The section 3 introduces the basic

notions about quadratic triples, quadric bundles and their degenerations, following [H-N-1] and [H-N-2]. We introduce the étale cohomological invariants of nondegenerate quadratic triples and quadric bundles in section 4, and explain why the arguments in [Bh] extend from base a field k to an excellent Dedekind domain S, so as to show that the functorial characteristic classes and cohomological invariants on quasi-projective schemes over such a base S exactly form the rings $H^*(BGO_{n,S}, \mathbb{F}_2)$ and $PH^*(BGO_{n,S}, \mathbb{F}_2)$. The section 5 is devoted to the proof of our main theorem (Theorem 11). The idea of the proof is to introduce a universal stack $\mathcal{M}$ for at most mildly degenerate quadratic triples (section 5.2) and to simulate a tubular neighbourhood of the degeneration locus $\mathcal{D}$ in $\mathcal{M}$, up to $\mathbf{A}^1$-homotopy, by another algebraic stack (introduced in section 5.3) which is the stackification of the 'standard model' for degenerating quadrics which was earlier introduced in [H-N-2].

Dedication The question of the behaviour of quadric invariants under degeneration was originally posed by M.S. Narasimhan in 1985 in order to test the rationality of some unirational moduli spaces. This was motivated by his cohomological criterion for rationality (which he discovered in the late 1960's), which says that the third integral cohomology of a nonsingular complex projective variety is torsion free if the variety is rational. The attempt to test of this criterion for the Narasimhan-Ramanan desingularization of moduli [Na-Ra], which has a natural degenerating family of conics on it, led to the results described in [N-1] and [N-2]. We are happy to dedicate this paper to Professor Narasimhan on his 80th birthday, for providing continued encouragement and inspiration over the years.

2 Purity and Gysin boundary maps

Regular pairs of schemes

A **regular scheme** will as usual mean a locally noetherian scheme all whose local rings are regular. We will work over a regular base scheme S on which 2 is invertible. All schemes, algebraic spaces and algebraic stacks and all morphisms will be assumed to be over S, and the structure morphisms to S will be assumed to be quasi-compact. Important extreme examples are when $S = \operatorname{Spec}\mathbb{Z}[1/2]$ or when $S = \operatorname{Spec} k$ for an algebraically closed field of characteristic $\neq 2$.

A **regular pair** (X, Y) **of schemes of pure codimension** 1 (or simply a 'regular pair', as the codimension will always be assumed to be 1 in what follows) consists of a regular scheme X and a closed subscheme Y which is regular and of codimension 1 in X at all points of Y. In particular, Y is an effective Cartier divisor in X. A special class of regular pairs are the **smooth pairs over the base** S, where S is a given regular scheme, and X and Y are smooth S-schemes with Y a closed subscheme of X, which we will assume to have codimension 1. An example of a regular pair is where $X = \operatorname{Spec}\mathbb{Z}[1/2]$ and $Y = \operatorname{Spec}\mathbb{F}_p$ where p is an odd prime. Note that in this example X is smooth over the base $S = \operatorname{Spec}\mathbb{Z}[1/2]$ but Y is not, so this regular pair is not a smooth pair over S. Regular or smooth pairs of algebraic spaces have a similar definition.

Recall that if (X, Y) is a regular pair of algebraic spaces of codimension 1 such that 2 is invertible on X, then the absolute purity theorem of Gabber (originally conjectured by Grothendieck – see [Fu] for Gabber's proof) gives us the following.

1 Gabber's absolute purity theorem. Under the above hypothesis, the local cohomology sheaves $\underline{H}^r_Y(\mathbb{F}_{2,X})$ on X are given by

$$\underline{H}^r_Y(\mathbb{F}_{2,X}) = \begin{cases} 0 & \text{for } r \neq 2, \\ i_*\mathbb{F}_{2,Y} & \text{for } r = 2. \end{cases}$$

By the resulting degenerateness of the local-global spectral sequence, this gives the natural identifications

$$H^r_Y(X, \mathbb{F}_2) = H^{r-2}(X, \underline{H}^2_Y(\mathbb{F}_{2,X})) = H^{r-2}(Y, \mathbb{F}_2).$$

If moreover Y is irreducible, then under the above isomorphism the **fundamental class** $s_{Y/X} \in H^2_Y(X, \mathbb{F}_2)$ of Y in X corresponds to the generating section in $H^0(X, \underline{H}^2_Y(\mathbb{F}_{2,X}))$.

Under the isomorphism $H^r_Y(X, \mathbb{F}_2) \cong H^{r-2}(Y, \mathbb{F}_2)$, the long exact sequence for cohomology with supports Y becomes the long exact Gysin sequence

$$\ldots \to H^r(X, \mathbb{F}_2) \to H^r(X - Y, \mathbb{F}_2) \xrightarrow{\partial} H^{r-1}(Y, \mathbb{F}_2) \to H^{r+1}(X, \mathbb{F}_2) \to \ldots$$

for the pair (X, Y). The maps $\partial : H^r(X - Y, \mathbb{F}_2) \to H^{r-1}(Y, \mathbb{F}_2)$ are the **Gysin boundary maps**.

Cohomology and local cohomology for algebraic stacks

We largely follow the terminology of Laumon and Moret-Bailly [L-MB] for algebraic stacks. Recall that we work over a regular base scheme S on which 2 is invertible, and all schemes, algebraic spaces and algebraic stacks are assumed to be quasi-compact over S. The word 'stack' will signify such an algebraic stack, and morphisms will always be over S.

Let $\mathit{Lis}\text{-}\mathit{Et}(\mathfrak{X})$ denote the smooth-étale site of $\mathfrak{X}$, and let $\mathfrak{X}_{\mathit{lis}\text{-}\mathit{et}}$ be the resulting topos, which we will denote simply by $\mathfrak{X}$. All sheaves and cohomology for $\mathfrak{X}$ will be in the smooth-étale topology. The sheaf of main interest to us is the constant sheaf $\mathbb{F}_{2,\mathfrak{X}}$ in $\mathfrak{X}_{\mathit{lis}\text{-}\mathit{et}}$. This sheaf is Cartesian and representable. We will simply denote it by $\mathbb{F}_2$. We denote by $\mathit{Mod}_{\mathit{cart}}(\mathfrak{X}, \mathbb{F}_2)$ the abelian category of Cartesian sheaves of $\mathbb{F}_2$-modules on $\mathfrak{X}$, and by $D^+_{\mathit{cart}}(\mathfrak{X}, \mathbb{F}_2)$ the derived category of cohomologically Cartesian bounded below complexes of sheaves of $\mathbb{F}_2$-modules on $\mathfrak{X}$. For an algebraic space X, the inclusion functor $\mathit{Et}(X) \hookrightarrow \mathit{Lis}\text{-}\mathit{Et}(X)$ from the étale to the smooth site induces a geometric morphism of topoi $X_{\mathit{lis}\text{-}\mathit{et}} \to X_{et}$ under which we get an equivalence between X_{et} and the subcategory of Cartesian sheaves in $X_{\mathit{lis}\text{-}\mathit{et}}$ (see [L-MB] 12.2.3). This induces an equivalence between $D^+(X_{et}, \mathbb{F}_2)$ and $D^+_{\mathit{cart}}(X_{\mathit{lis}\text{-}\mathit{et}}, \mathbb{F}_2)$. Consequently, étale cohomology (or hypercohomology) on X_{et} where X is an algebraic space may be regarded as the smooth-étale cohomology (or hypercohomology) of the corresponding Cartesian object on $X_{\mathit{lis}\text{-}\mathit{et}}$.

It is known due to Behrend and Gabber that a 1-morphism $f : \mathfrak{X}' \to \mathfrak{X}$ of stacks does not functorially induce a geometric morphism of topoi from $\mathfrak{X}'_{lis\text{-}et}$ to $\mathfrak{X}_{lis\text{-}et}$. However, Olsson showed in [O] how to functorially associate to f a pull-back functor

$$f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$$

of triangulated categories. The facts about f^* that we need are only a small part of the theory of six operations for algebraic stacks developed by Behrend [Be], Olsson [O], Laszlo and Olsson [L-O], and other authors.

Notation: The functor $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$ is actually called f^{-1} in [O]. Our notation f^* for it follows the section 4.3 of [L-O].

We now briefly recall the construction of f^* from [O]. Let $X \to \mathfrak{X}$ be a smooth atlas where X is an algebraic space over S, and let $X^+_\bullet = cosk^+_0(X/\mathfrak{X})$ be its nerve regarded as a strictly simplicial algebraic space, where X_n is the fiber product of $n+1$ copies of X over $\mathfrak{X}$. The word 'strict' means that only the morphisms $X_n \to X_m$ in $cosk_0(X/\mathfrak{X})$ which correspond to strictly monotonic maps $[m] \to [n]$ (and which are therefore smooth morphisms) are retained as part of the structure of $X^+_\bullet$. As these morphisms are smooth, the topos $X^+_{\bullet,lis\text{-}et}$ makes sense. Let $X^+_{\bullet,et}$ be the corresponding topos with étale topology.

There are natural geometric morphisms of topoi $\pi : X^+_{\bullet,lis\text{-}et} \to \mathfrak{X}_{lis\text{-}et}$ and $\epsilon : X^+_{\bullet,lis\text{-}et} \to X^+_{\bullet,et}$. Using the theory of cohomological descent developed in [SGA 4] and a result of Gabber, Olsson proves (Theorem 4.7 of [O]) that the induced functors $\pi^* : D^+_{cart}(\mathfrak{X}_{lis\text{-}et}) \to D^+_{cart}(X^+_{\bullet,lis\text{-}et})$ and $\epsilon^* : D^+_{cart}(X^+_{\bullet,et}) \to D^+_{cart}(X^+_{\bullet,lis\text{-}et})$ are equivalences of triangulated categories, with quasi-inverses respectively $R\pi_*$ and ϵ_*.

Note: The Cartesian conditions for $\mathfrak{X}_{lis\text{-}et}$ and for $X^+_{\bullet,et}$ are meant respectively in the sheaf sense and in the strictly simplicial sense, while the Cartesian condition for $X^+_{\bullet,lis\text{-}et}$ is meant to be simultaneously satisfied in both the sheaf and the strictly simplicial senses.

The composite $\epsilon_* \circ \pi^*$ defines an equivalence of triangulated categories

$$\psi : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(X^+_{\bullet,et}, \mathbb{F}_2).$$

Given a 1-morphism of S-stacks $f : \mathfrak{X}' \to \mathfrak{X}$, let $\mathfrak{X}'' = \mathfrak{X}' \times_{\mathfrak{X}} X$ (which is a stack), and let $X' \to \mathfrak{X}''$ be a smooth atlas where X' is an algebraic space. Then we can define a similar equivalence of triangulated categories $\psi' : D^+_{cart}(\mathfrak{X}', \mathbb{F}_2) \to D^+_{cart}(X'^+_\bullet, \mathbb{F}_2)$. The induced morphism $f_\bullet : X'^+_\bullet \to X^+_\bullet$ actually gives a geometric morphism of topoi

$$(f^*_\bullet, f_{\bullet,*}) : X'^+_{\bullet,et} \to X^+_{\bullet,et}$$

so it induces a functor of triangulated categories

$$f^*_\bullet : D^+_{cart}(X^+_{\bullet,et}, \mathbb{F}_2) \to D^+_{cart}(X'^+_\bullet, \mathbb{F}_2).$$

Finally, Olsson defines (see 9.16 of [O]) the functor $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$ as the composite

$$D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \stackrel{\psi}{\to} D^+_{cart}(X^+_{\bullet,et}, \mathbb{F}_2) \stackrel{f^*_\bullet}{\to} D^+_{cart}(X'^+_{\bullet,et}, \mathbb{F}_2) \stackrel{\psi'^{-1}}{\to} D^+_{cart}(\mathfrak{X}', \mathbb{F}_2).$$

It is shown in [O] that the above functor f^* is left adjoint to the functor $Rf_* : D^+_{cart}(\mathfrak{X}', \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$. Hence it is well-defined, independent of the choices of the smooth atlases $X \to \mathfrak{X}$ and $X' \to \mathfrak{X}''$. Moreover, it is functorial in f, that is, given $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$, we have $(g \circ f)^* = f^* \circ g^*$.

Note In the special case where the 1-morphism $f : \mathfrak{X}' \to \mathfrak{X}$ is smooth (but not necessarily representable), the pullback $f^* : \mathfrak{X}_{lis\text{-}et} \to \mathfrak{X}'_{lis\text{-}et}$ on sheaves of sets is indeed (left) exact, and so $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$ can be defined directly. As one would desire, the direct definition of $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$ when f is smooth coincides with the above indirect general definition.

For any $f : \mathfrak{X}' \to \mathfrak{X}$, with the above definition of the functor $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$, we have $f^*\mathbb{F}_{2,\mathfrak{X}} = \mathbb{F}_{2,\mathfrak{X}'}$. Hence for any F in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$, the functor $f^* : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$ induces a pull back homomorphism f^* on hypercohomology

$$\mathbb{H}^i(\mathfrak{X}, F) = Hom_{D^+_{cart}(\mathfrak{X},\mathbb{F}_2)}(\mathbb{F}_2, F[i]) \to Hom_{D^+_{cart}(\mathfrak{X}',\mathbb{F}_2)}(\mathbb{F}_2, f^*F[i]) = \mathbb{H}^i(\mathfrak{X}', f^*F).$$

In the important special case where $F = \mathbb{F}_2$, we thus get homomorphisms

$$f^* : H^i(\mathfrak{X}, \mathbb{F}_2) \to H^i(\mathfrak{X}', \mathbb{F}_2).$$

If $i : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ is a closed substack of an algebraic stack $\mathfrak{X}$ over S, and if F is a sheaf in $\mathfrak{X}_{lis\text{-}et}$, then recall that the (0)-th local cohomology sheaf $\underline{H}^0_{\mathfrak{Y}}(F) = i_*i^!(F)$ can be directly defined as follows. For any U in $Lis\text{-}Et(\mathfrak{X})$, let $V \subset U$ be the pullback of $\mathfrak{Y}$. Then $\underline{H}^0_{\mathfrak{Y}}(F)(U)$ is defined to be the kernel of the restriction map $F(U) \to F(U - V)$. The functor $F \mapsto \underline{H}^0_{\mathfrak{Y}}(F)$ is left exact, and so we have its derived functor $i_*Ri^! : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$. (*Note*: It is enough for our purpose to directly define $i_*Ri^!$, without first defining $Ri^!$ and then composing it with i_*.)

If $j : \mathfrak{X} - \mathfrak{Y} \hookrightarrow \mathfrak{X}$ is the complementary open inclusion, then for any F in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$ we have a functorial exact triangle $(i_*Ri^!\mathbb{F}_{2,\mathfrak{X}} \to \mathbb{F}_{2,\mathfrak{X}} \to j_*j^*\mathbb{F}_{2,\mathfrak{X}} \to)$ in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$, which corresponds under the equivalence $\psi : D^+_{cart}(\mathfrak{X}, \mathbb{F}_2) \to D^+_{cart}(X^+_{\bullet,et}, \mathbb{F}_2)$ to the exact triangle $(i_{\bullet,*}Ri^!_{\bullet}\mathbb{F}_{2,X^+_{\bullet,et}} \to \mathbb{F}_{2,X^+_{\bullet,et}} \to j_{\bullet,*}j^*_{\bullet}\mathbb{F}_{2,X^+_{\bullet,et}} \to)$ in $D^+_{cart}(X^+_{\bullet,et}, \mathbb{F}_2)$. Here, $Y \subset X$ is the pullback of $\mathfrak{Y}$ under the atlas $X \to \mathfrak{X}$, $i_\bullet : Y^+_\bullet \hookrightarrow X^+_\bullet$ is the corresponding closed strictly simplicial subspace and $j_\bullet : X^+_\bullet - Y^+_\bullet \hookrightarrow X^+_\bullet$ is the complementary open inclusion.

The local cohomology groups are defined in terms of $i_*Ri^!$ as the hypercohomologies

$$H^r_{\mathfrak{Y}}(\mathfrak{X}, \mathbb{F}_2) = \mathbb{H}^r(\mathfrak{X}, i_*Ri^!\mathbb{F}_2).$$

Let $i : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ (resp. $i' : \mathfrak{Y}' \hookrightarrow \mathfrak{X}'$) be closed substacks, such that the closed substack $f^{-1}(\mathfrak{Y}) \subset \mathfrak{X}'$ has the same support as $\mathfrak{Y}'$. We have a natural map

$$f^*i_*Ri^!\mathbb{F}_{2,\mathfrak{X}} \to i'_*Ri'^*\mathbb{F}_{2,\mathfrak{X}'}$$

in $D^+_{cart}(\mathfrak{X}')$ (see below), which at the level of hypercohomologies gives a pull back homomorphism on local cohomologies

$$f^* : H^i_{\mathfrak{Y}}(\mathfrak{X}, \mathbb{F}_2) \to H^i_{\mathfrak{Y}'}(\mathfrak{X}', \mathbb{F}_2).$$

The homomorphism $f^*i_*Ri^!\mathbb{F}_{2,\mathfrak{X}} \to i'_*Ri'^*\mathbb{F}_{2,\mathfrak{X}'}$ has the following strictly simplicial construction. Choosing atlases as above, let $i_\bullet : Y_\bullet^+ \subset X_\bullet^+$ and $i'_\bullet : Y_\bullet'^+ \subset X_\bullet'^+$ be the corresponding inverse images. Note that f gives $f_\bullet : X_\bullet'^+ \to X_\bullet^+$ under which the support of $f_\bullet^{-1}(Y_\bullet^+)$ is the support of $Y_\bullet'^+$. Then $f_\bullet$ induces a commutative diagram (morphism of exact triangles) in $D^+_{cart}(X'^+_{\bullet,et}, \mathbb{F}_2)$

$$\begin{array}{ccccccc}
f_\bullet^* i_{\bullet,*} Ri_\bullet^! \mathbb{F}_{2,X^+_{\bullet,et}} & \to & f_\bullet^* \mathbb{F}_{2,X^+_{\bullet,et}} & \to & f_\bullet^* j_{\bullet,*} j_\bullet^* \mathbb{F}_{2,X^+_{\bullet,et}} & \to \\
\downarrow & & \downarrow & & \downarrow & \\
i'_{\bullet,*} Ri'^!_\bullet \mathbb{F}_{2,X'^+_{\bullet,et}} & \to & \mathbb{F}_{2,X'^+_{\bullet,et}} & \to & j'_{\bullet,*} j'^*_\bullet \mathbb{F}_{2,X'^+_{\bullet,et}} & \to
\end{array}$$

Hence we get the following commutative diagram in $D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$

$$\begin{array}{ccccccc}
f^* i_* Ri^! \mathbb{F}_{2,\mathfrak{X}} & \to & f^* \mathbb{F}_{2,\mathfrak{X}} & \to & f^* j_* j^* \mathbb{F}_{2,\mathfrak{X}} & \to \\
\downarrow & & \downarrow & & \downarrow & \\
i'_* Ri'^! \mathbb{F}_{2,\mathfrak{X}'} & \to & \mathbb{F}_{2,\mathfrak{X}'} & \to & j'_* j'^* \mathbb{F}_{2,\mathfrak{X}'} & \to
\end{array}$$

Remark 2 To check that a homomorphism in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$ is zero (or to check that a diagram commutes in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$), it is *not enough* to check that its pull back is zero in $D^+(X_{et}, \mathbb{F}_2)$ for an atlas $X \to \mathfrak{X}$ – rather, one needs to check that it is zero in the derived category of the strictly simplicial topos $X^+_{\bullet,et}$, so that cohomological descent can be employed. However, if it is a homomorphism (or a diagram) coming from $Mod_{cart}(\mathfrak{X}, \mathbb{F}_2)$, to check that it is zero (or commutes) in $D^+_{cart}(\mathfrak{X}, \mathbb{F}_2)$, it is enough to check that its pull back to $Mod(X_{et}, \mathbb{F}_2)$ is zero (or commutes).

Remark 3 By taking hypercohomologies, the above morphism of exact triangles gives a commutative diagram

$$\begin{array}{ccccccc}
H^i(\mathfrak{X}, \mathbb{F}_2) & \to & H^i(\mathfrak{X} - \mathfrak{Y}, \mathbb{F}_2) & \stackrel{d}{\to} & H^{i+1}_{\mathfrak{Y}}(\mathfrak{X}, \mathbb{F}_2) & \to & H^{i+1}(\mathfrak{X}, \mathbb{F}_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^i(\mathfrak{X}', \mathbb{F}_2) & \to & H^i(\mathfrak{X}' - \mathfrak{Y}', \mathbb{F}_2) & \stackrel{d}{\to} & H^{i+1}_{\mathfrak{Y}'}(\mathfrak{X}', \mathbb{F}_2) & \to & H^{i+1}(\mathfrak{X}', \mathbb{F}_2)
\end{array}$$

Regular pairs of algebraic stacks

Recall that an algebraic stack $\mathfrak{X}$ is said to be a **regular stack** if for some (hence for every) smooth atlas $X \to \mathfrak{X}$ where X is a scheme (or an algebraic space), the total space X is regular. A **regular pair** $(\mathfrak{X}, \mathfrak{Y})$ **of stacks** will mean a closed embedding $i : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ such that both $\mathfrak{X}$ and $\mathfrak{Y}$ are regular stacks, and $\mathfrak{Y}$ has pure codimension 1. Certain crucial examples of such pairs for us, namely the pairs $(\mathcal{M}, \mathcal{D})$ and $(\mathcal{N}, \mathcal{Z})$ that we introduce later, are in fact **smooth pairs of** S**-stacks**, that is, the base scheme S is regular and the structure morphisms $\mathfrak{X} \to S$ and $\mathfrak{Y} \to S$ are smooth.

4 (Absolute purity theorem for algebraic stacks.) With all the above preparation, it follows from the absolute purity theorem for schemes that the absolute purity theorem (Statement 1) holds also for regular pairs of stacks $(\mathfrak{X}, \mathfrak{Y})$ of pure codimension 1 (see [L-O] section 4.10). In particular, we get a long exact Gysin sequence as in Statement 1.

Lemma 5 *Let $(\mathfrak{X}, \mathfrak{Y})$ and $(\mathfrak{X}', \mathfrak{Y}')$ be regular pairs of algebraic stacks of pure codimension 1 such that $\mathfrak{Y}$ and $\mathfrak{Y}'$ are connected, and 2 is invertible on $\mathfrak{X}$ and $\mathfrak{X}'$. Let $f : \mathfrak{X}' \to \mathfrak{X}$ be a 1-morphism such that the closed substack $\mathfrak{Y}'' = \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{Y}$ of $\mathfrak{Y}$ is equal to $m\mathfrak{Y}'$ for some integer $m \geq 1$ (that is, its ideal sheaf $I_{\mathfrak{Y}''}$ equals $I_{\mathfrak{Y}'}^m$). Then the following statements hold.*

(1) Under $f^* : H^2_{\mathfrak{Y}}(\mathfrak{X}, \mathbb{F}_2) \to H^2_{\mathfrak{Y}'}(\mathfrak{X}', \mathbb{F}_2)$, the fundamental classes $s_{\mathfrak{Y}/\mathfrak{X}}$ and $s_{\mathfrak{Y}'/\mathfrak{X}'}$ are related by $s_{\mathfrak{Y}/\mathfrak{X}} \mapsto m \cdot s_{\mathfrak{Y}'/\mathfrak{X}'}$.

(2) The following diagram commutes

$$\begin{array}{ccccccc}
H^i(\mathfrak{X}, \mathbb{F}_2) & \to & H^i(\mathfrak{X} - \mathfrak{Y}, \mathbb{F}_2) & \stackrel{\partial}{\to} & H^{i-1}(\mathfrak{Y}, \mathbb{F}_2) & \to & H^{i+1}(\mathfrak{X}, \mathbb{F}_2) \\
\downarrow {\scriptstyle f^*} & & \downarrow {\scriptstyle f^*_{\mathfrak{X}'-\mathfrak{Y}'}} & & \downarrow {\scriptstyle mf^*_{\mathfrak{Y}'}} & & \downarrow {\scriptstyle f^*} \\
H^i(\mathfrak{X}', \mathbb{F}_2) & \to & H^i(\mathfrak{X}' - \mathfrak{Y}', \mathbb{F}_2) & \stackrel{\partial'}{\to} & H^{i-1}(\mathfrak{Y}', \mathbb{F}_2) & \to & H^{i+1}(\mathfrak{X}', \mathbb{F}_2)
\end{array}$$

where the rows are the respective long exact Gysin sequences. In particular,

$$\partial \circ f^*_{\mathfrak{X}'-\mathfrak{Y}'} = m \cdot f^*_{\mathfrak{Y}} \circ \partial'.$$

Proof The statement (1) for smooth pairs of schemes is due to Grothendieck (see [SGA $4\frac{1}{2}$] page 138), who used the purity theorem for smooth pairs. If we instead use the absolute purity theorem, the same proof applies to prove the statement (1) for regular pairs (X, Y) of schemes or algebraic spaces.

Now consider the special case when (X, Y) and (X', Y') are regular pairs of algebraic spaces, and $f : X' \to X$ is a morphism for which $f^{-1}(Y) = mY'$. As $s_{Y/X} \mapsto m \cdot s_{Y'/X'}$, the following diagram commutes, where the horizontal isomorphisms come from purity, and where the right hand vertical map is multiplication by m.

$$\begin{array}{ccc}
f^*i_*Ri^!\mathbb{F}_{2,X} & \stackrel{\sim}{\to} & f^*i_*\mathbb{F}_{2,Y}[-2] \\
\downarrow & & \downarrow {\scriptstyle m} \\
i'_*Ri'^!\mathbb{F}_{2,X'} & \stackrel{\sim}{\to} & i_*\mathbb{F}_{2,Y'}[-2]
\end{array}$$

Note that the above is actually (the shift by -2 of) a commutative diagram coming from $Mod_{cart}(X', \mathbb{F}_2)$. Hence it follows by Remark 2 that the following diagram commutes in $D^+_{cart}(\mathfrak{X}', \mathbb{F}_2)$.

$$\begin{array}{ccc}
f^*i_*Ri^!\mathbb{F}_{2,\mathfrak{X}} & \stackrel{\sim}{\to} & f^*i_*\mathbb{F}_{2,\mathfrak{Y}}[-2] \\
\downarrow & & \downarrow {\scriptstyle m} \\
i'_*Ri'^!\mathbb{F}_{2,\mathfrak{X}'} & \stackrel{\sim}{\to} & i_*\mathbb{F}_{2,\mathfrak{Y}'}[-2]
\end{array}$$

The statement (1) follows for stacks by applying hypercohomology functors $\mathbb{H}^2$ to the above diagram and chasing the image of the fundamental class $s_{\mathfrak{Y}/\mathfrak{X}}$ in the resulting diagram.

Applying $(i+1)$-th hypercohomology functors to the above diagram, it follows that the following diagram commutes.

$$\begin{array}{ccc} H^{i+1}_{\mathfrak{Y}}(\mathfrak{X},\mathbb{F}_2) & \stackrel{\sim}{\to} & H^{i-1}(\mathfrak{Y},\mathbb{F}_2) \\ {\scriptstyle f^*}\downarrow & & \downarrow {\scriptstyle m\, f^*_{\mathfrak{Y}'}} \\ H^{i+1}_{\mathfrak{Y}'}(\mathfrak{X}',\mathbb{F}_2) & \stackrel{\sim}{\to} & H^{i-1}(\mathfrak{Y}',\mathbb{F}_2) \end{array}$$

By combining the above commutative diagram with the one given by Remark 3, the statement (2) follows. $\square$

Remark 6 The conclusions of Lemma 5.(2) remains true even if in the hypothesis of the lemma the stacks Y and Y' are not assumed to be connected and the multiplicity m is allowed to vary as a locally constant function over Y'. To see this, note that if Y_i are the connected components of Y and Y'_j are the connected components of Y', and if Y'_{j_0} maps into Y_{i_0}, then we can apply the lemma to the pairs $(X - \cup_{i\neq i_0} Y_i,\, Y_{i_0})$ and $(X' - \cup_{j\neq j_0} Y'_j,\, Y'_{j_0})$, and conclude by excision.

Remark 7 Any smooth atlas $X \to \mathfrak{X}$ of an S-stack functorially gives a smooth atlas $X \times \mathbf{A}^1 \to \mathfrak{X} \times \mathbf{A}^1$ where $\mathbf{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$. Using these atlases and the corresponding property for algebraic spaces, it follows from the definition of the pullback map on cohomologies of stacks that if $f, g \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ are regular functions with graphs $\gamma_f, \gamma_g : \mathfrak{X} \to \mathfrak{X} \times \mathbf{A}^1$, then we have $\gamma_f^* = \gamma_g^* : H^*(\mathfrak{X} \times \mathbf{A}^1, \mathbb{F}_2) \to H^*(\mathfrak{X}, \mathbb{F}_2)$.

3 Quadric bundles and their degenerations

Quadratic triples and quadric bundles

Let S be a scheme over $\mathbb{Z}[1/2]$. A **quadratic triple** $T = (E, L, b)$ on an S-scheme X by definition consists of a bilinear form b on a vector bundle E on X, taking values in a line bundle L on X, such that b is nowhere zero on X, that is, b_x is not identically zero on any fiber E_x. We call $\operatorname{rank}(E)$ (which we assume to be globally constant) the **dimension** of the triple T and the $\operatorname{rank}(b_x)$ of the quadratic form b_x on the fiber $E_x = k(x) \otimes_{\mathcal{O}_X} E$ as the **rank** of T at $x \in X$. We will say that a quadratic triple $T = (E, L, b)$ on X is **nondegenerate** if at each $x \in X$ we have $\operatorname{rank}(b_x) = \operatorname{rank}(E)$, equivalently, the $\mathcal{O}_X$-linear map $b : E \to L \otimes_{\mathcal{O}_X} E^*$ is an isomorphism.

Given a triple $T = (E, L, b)$ and a line bundle K on X, the triple $T \otimes K$ is defined to be $(E \otimes K, L \otimes K^{\otimes 2}, b \otimes 1_K)$. We say that two triples $T = (E, L, b)$ and $T' = (E', L', b')$ on X are **equivalent triples** if there exists a line bundle K on X such that $T \otimes K$ is isomorphic to T'. A **quadric bundle** Q over X is by

definition an equivalence class $[E, L, b]$ of triples such that for each $x \in X$ we have $b_x \neq 0$. As b_x is not identically zero on any fiber E_x by assumption, we see that $[E, L, b]$ defines a closed subscheme $C \subset P(E) = \mathrm{Proj}\,\mathrm{Sym}(E^*)$ by the vanishing of the quadratic form corresponding to b, which is flat over X and whose each fiber is a quadric hypersurface $C_x \subset P(E_x)$. Conversely the equivalence class $Q = [E, L, b]$ can be recovered from a closed subscheme $C \subset P(E)$ in a banal projective bundle, such that C is flat over X and every schematic fiber is a quadric hypersurface.

Quadratic triples and quadric bundles over algebraic stacks: In the above definitions, the base space for quadratic triples and quadric bundles was taken to be an S-scheme X. But the same definitions work equally well when X is an algebraic stack over S. In what follows, the base spaces for our quadratic triples or quadric bundles will be assumed to be algebraic stacks over S, unless otherwise indicated.

Minimally degenerate quadric bundles, associated nondegenerate triples

The **discriminant** $\det(T)$ of a quadratic triple $T = (E, L, b)$ on X is defined by

$$\det(T) = \det(b) \in \Gamma(X, L^{\mathrm{rank}(E)} \otimes \det(E)^{-2}).$$

A quadratic triple T is nondegenerate if and only if $\det(T)$ is nowhere vanishing. Note that

$$\det(T) = \det(T \otimes K)$$

for any line bundle K on X, so for any quadric bundle $Q = [E, L, b]$. This enables us to define **the discriminant** $\det(Q)$ **of a quadric bundle** to be $\det(T)$ for any representative T.

A quadratic triple (E, L, b) on base Y is called a **minimally degenerate triple** if (i) $\det(b) = 0 \in \Gamma(Y, L^{\mathrm{rank}(E)} \otimes \det(E)^{-2})$ and (ii) $\mathrm{rank}(b_y) = n - 1$ for all $y \in Y$. (Note that if Y is reduced then (ii) implies (i), but not otherwise.) The equivalence class $Q = [E, L, b]$ of such a triple will be called a **minimally degenerate quadric bundle** (this is well defined, independent of the choice of a representative (E, L, b) of Q). If we regard b as an $\mathcal{O}_Y$-linear homomorphism $b : E \to L \otimes E^*$, then $\ker(b)$ is a line bundle, the quotient $\overline{E} = E/\ker(b)$ is a vector bundle of rank equal to $\mathrm{rank}(E) - 1$, and b induces an L-valued quadratic form $\overline{b}$ on $\overline{E}$, giving a nondegenerate quadratic triple $(\overline{E}, L, \overline{b})$.

The next lemmas follows from a simple computation which we omit.

Lemma 8 *If $Q = [E, L, b]$ is a minimally degenerate quadric bundle on Y, then the nondegenerate quadratic triple*

$$\mathbb{T}^Q = (\overline{E}, L, \overline{b}) \otimes \ker(b)^{-1}$$

is well defined, that is, it remains unaltered if the representative (E, L, b) is replaced by $(E, L, b) \otimes K$ for any line bundle K on Y. □

We will call the above quadratic triple $\mathbb{T}^Q = (\overline{E}, L, \overline{b}) \otimes \ker(b)^{-1}$ (which was originally introduced in [H-N-2]) the **associated nondegenerate quadratic triple** to the minimally degenerate quadric bundle Q.

Mildly degenerating triples on a regular pair (X, Y)

A **mildly degenerating triple** (E, L, b) on a regular pair (X, Y) of stacks is a quadratic triple $T = (E, L, b)$ on X such that $T|_{X-Y}$ is nondegenerate while $T|_Y$ is minimally degenerate, that is, $\mathrm{rank}_x(b) = \mathrm{rank}(E)$ for each $x \in X - Y$ while $\mathrm{rank}_y(b) = \mathrm{rank}(E) - 1$ for each $y \in Y$. As such a Y is reduced, the discriminant $\det(T)$ vanishes on Y.

Given a mildly degenerating triple $T = (E, L, b)$ on a regular pair (X, Y) of schemes, the vanishing multiplicity of the discriminant $\det(T)$ along the divisor Y defines a function $\nu_Y(T) : |Y| \to \mathbb{Z}_{\geq 0}$ on the set $|Y|$ of connected components of Y. This function will be called the **degeneration multiplicity** of the triple. If (X, Y) is a regular pair and $m : |Y| \to \mathbb{Z}_{\geq 0}$ is a continuous function, then we get a new ideal sheaf $I_Y^m \subset \mathcal{O}_X$, and we denote by $mY \subset X$ the corresponding closed subscheme. Note that if $m = 0$ then $mY = \emptyset$ and if $m = 1$ then $mY = Y$. The above has an obvious interpretation also for mildly degenerating triple on a regular pair of stacks.

Remark 9 In particular, if (X, Y) is a regular pair and if Q is a mildly degenerating quadric bundle on (X, Y), then the restriction Q_Y is a minimally degenerate quadric bundle on Y, which gives us an associated nondegenerate quadratic triple $\mathbb{T}^{Q_Y}$.

4 Cohomological invariants

The group schemes GO_n and stacks BGO_n

The **general orthogonal group-scheme** GO_n over $\mathbb{Z}[1/2]$ is the closed sub-group scheme of GL_n over $\mathbb{Z}[1/2]$, whose R-valued points for any $\mathbb{Z}[1/2]$-algebra R (that is, a commutative ring R in which 2 is invertible) are all the $n \times n$ matrices g over R such that ${}^t g g = \lambda I$ for some invertible element $\lambda \in R^\times$. We have a surjective homomorphism $\sigma : GO_n \to \mathbb{G}_m$ of group schemes defined on valued points by $g \mapsto \lambda$ where ${}^t g g = \lambda I$. This has kernel O_n, the orthogonal group scheme over $\mathbb{Z}[1/2]$. As both O_n and $\mathbb{G}_m$ are smooth over $\mathbb{Z}[1/2]$, it follows that GO_n is also smooth over $\mathbb{Z}[1/2]$. In fact, it can be seen that GO_n is a reductive group scheme over $\mathbb{Z}[1/2]$.

For any scheme S over $\mathbb{Z}[1/2]$, the group-scheme $GO_{n,S}$ is obtained by base change, and has a similar description in terms of R-valued points for affine schemes $\mathrm{Spec}\, R \to S$. The group schemes $GL_{n,S}$, $\mathbb{G}_{m,S}$, $GO_{n,S}$, $O_{n,S}$, etc. will be simply denoted by GL_n, $\mathbb{G}_m$, GO_n, O_n, etc. for simplicity of notation.

When $n = 2m + 1$ is odd, the homomorphism $\mathbb{G}_m \times SO_{2m+1} \to GO_{2m+1}$ defined in terms of valued points by $(\lambda, g) \mapsto \lambda g$ is an isomorphism. This induces a 1-isomorphism of algebraic stacks $B\mathbb{G}_m \times BSO_{2m+1} \to BGO_{2m+1}$.

If X is a scheme over S, then a principal GO_n-bundle P on X, locally trivial in the étale topology, is equivalent to a nondegenerate quadratic triple (E, L, b). To any such P, we functorially associate the vector bundle E given by the defining representation $GO_n \hookrightarrow GL_n$, and L given by the character $\sigma : GO_n \to \mathbb{G}_m$ defined above. The standard bilinear form $\sum x_i y_i$ defines $b : E \otimes E \to L$. This defines a 1-isomorphism of stacks from the algebraic stack BGO_n to the algebraic stack $\mathcal{M} - \mathcal{D}$ of nondegenerate quadratic triples that is introduced later. The inverse 1-morphism has an obvious construction using the Gram-Schmidt orthogonalization process. The principal bundle P associated to any (E, L, b) is étale locally trivial as the process of Gram-Schmidt orthogonalization requires taking square-roots of invertible regular functions, which is possible étale locally as 2 is invertible over X.

Similarly, the stack BO_n can be identified with the stack of **nondegenerate quadratic pairs**, where such a pair (E, q) on an S-scheme X is just a nondegenerate triple $(E, \mathcal{O}_X, q)$.

On the algebraic stack BGO_n, we have a universal nondegenerate quadratic triple (E_n, L_n, b_n). Let L_n also denote the total space of the line bundle L_n, so that L_n is again an algebraic stack with a projection $\pi : L_n \to BGO_n$ and a zero section $BGO_n \to L_n$, which are 1-morphisms of algebraic stacks. The stack L_n can be obtained from BGO_n by a local construction in the sense of Laumon, Moret-Bailly [L-MB] Chapter 14.

Let $L_n^\times \subset L_n$ be the open substack which is the complement of the zero section $BGO_n \hookrightarrow L_n$. For any X over S, an object of $L_n^\times(X)$ is a tuple (E, L, b, s) where (E, L, b) is a nondegenerate quadratic triple on X, and $s \in \Gamma(X, L)$ is a nowhere vanishing section. Equivalently, $s : \mathcal{O}_X \to L$ is an isomorphism. Therefore there is a 1-morphism of stacks $BO_n \to L_n^\times$, which associates to a nondegenerate quadratic pair $(F, q : F \otimes F \to \mathcal{O}_X)$ on X the tuple $(F, \mathcal{O}_X, q, \mathrm{id}_{\mathcal{O}_X})$. In the reverse direction, we can associate to a tuple $(E, L, b, s : \mathcal{O}_X \to L)$ the quadratic pair $(E, s^{-1} \circ b)$, showing that the 1-morphism $BO_n \to L_n^\times$ is an isomorphism of algebraic stacks. Using this, we identify $L_n^\times$ with BO_n.

As GO_n is a smooth group scheme over S, the stack BGO_n is smooth over S, hence the total space L_n is also smooth over S. We regard $BGO_n \hookrightarrow L_n$ as the zero section, so that (L_n, BGO_n) is a smooth pair over S, in particular, it is a regular pair. The projection $\pi : L_n \to BGO_n$ induces an isomorphism $H^*(BGO_n, \mathbb{F}_2) \to H^*(L_n, \mathbb{F}_2)$. Identifying $L_n^\times$ with BO_n, the Gysin sequence for (L_n, BGO_n) becomes the following exact sequence.

$$\ldots \to H^*(BGO_n, \mathbb{F}_2) \to H^*(BO_n, \mathbb{F}_2) \xrightarrow{d_n} H^{*-1}(BGO_n, \mathbb{F}_2) \to H^{*+1}(BGO_n, \mathbb{F}_2) \to \ldots$$

Characteristic classes for nondegenerate triples and $H^*(BGO_n)$

Given a group-scheme G over a base S, let $|BG|$ denote the set-valued functor on $(QuasiProj_S)^{opp}$ which associates to any quasi-projective S-scheme X the set of all isomorphism classes of principal G-bundles on X. Assuming that 2 is invertible

on S, a **functorial characteristic class** for G will mean a natural transformation from $|BG|$ to the functor $H^*(-,\mathbb{F}_2)$ which associates to any X its étale cohomology $H^*(X,\mathbb{F}_2)$. All such functorial characteristic classes form a graded ring $CharClass^*_G$ under the ring operations on the various $H^*(X,\mathbb{F}_2)$.

If G is of finite-type and flat over S, then BG is an algebraic stack over S. Let $H^*(BG,\mathbb{F}_2)$ be its cohomology in the smooth-étale topology. We have a natural homomorphism

$$H^*(BG,\mathbb{F}_2) \to CharClass^*_G$$

of graded rings, which associates to any $\alpha \in H^*(BG,\mathbb{F}_2)$ the functorial characteristic class α' defined by $\alpha'(P) = \chi_P^*(\alpha)$ where $\chi_P : X \to BG$ is the classifying morphism of a G-bundle P on X.

When the base S is of the form $\operatorname{Spec} k$ for an algebraically closed field of characteristic $\neq 2$ and when G is reductive over k, it was shown in [Bh] that the homomorphism $H^*(BG,\mathbb{F}_2) \to CharClass^*_G$ is an isomorphism. We now revisit the argument in [Bh], to show it can be made to work over any excellent Dedekind domain on which 2 is invertible (for example, $\mathbb{Z}[1/2]$). This gives us the following.

Proposition 10 *Let G be a reductive group scheme over a base which is a field or an excellent Dedekind domain on which 2 is invertible. Then the natural homomorphism $H^*(BG,\mathbb{F}_2) \to CharClass^*_G$ is an isomorphism of graded rings.*

Proof (Sketch) The following three comments (1), (2) and (3) explain how the proof of the corresponding result (Proposition 3.1) in [Bh] can be modified so that it applies in our case.

(1) The Lemma 3.1 of [Bh] says that any closed reduced subscheme $Z \subset \mathbf{A}^n_k$ has a finite filtration by locally closed reduced subschemes $Z = Z_0 \supset Z_1 \supset \ldots \supset Z_\ell = \emptyset$ such that each $Z_i - Z_{i+1}$ is smooth over k. A modified statement holds for closed reduced subschemes $Z \subset \mathbf{A}^n_S$, with the condition '$Z_i - Z_{i+1}$ is smooth over k' replaced by the condition that $Z_i - Z_{i+1}$ is a regular scheme (we do not require $Z_i - Z_{i+1}$ to be smooth over S). Such a filtration exists because in any excellent scheme, regular points form an open subscheme (see Grothendieck [EGA-IV$_2$] page 215 statement 7.8.3.(iv)). An application of the absolute purity theorem of Gabber now completes the argument.

(2) The use of GIT for quotients for linear actions of reductive groups (in which which [Bh] follows Totaro [T]) is replaced by the use of Seshadri's version of GIT in [Se], which in particular works over the base S.

(3) Jouanolou's trick (in which [Bh] again follows Totaro [T]) remains valid over the base S (see [Ju]).

With the above modifications, the rest of the argument now works as in [Bh]. $\square$

Cohomological invariants for nondegenerate quadric bundles

Tensor product of triples and line bundles defines a 1-morphism of algebraic stacks

$$\mu : BGO_n \times B\mathbb{G}_m \to BGO_n$$

which sends $(T, K) \mapsto T \otimes K$ where T is a quadratic triple on X and K is a line bundle on X. Following Toda [Td] as in [H-N-2], we say that an element $\alpha \in H^*(BGO_n, \mathbb{F}_2)$ is a **primitive class** if under the cohomology homomorphisms induced by μ and by the projection $p_1 : BGO_n \times B\mathbb{G}_m \to BGO_n$, we have

$$\mu^*(\alpha) = p_1^*(\alpha).$$

The primitive classes form a subring $PH^*(BGO_n, \mathbb{F}_2) \subset H^*(BGO_n, \mathbb{F}_2)$. Over a field or an excellent Dedekind domain S on which 2 is invertible, the arguments used for proving that the natural homomorphism $H^*(BG, \mathbb{F}_2) \to CharClass_G^*$ is an isomorphism can be used to see that elements $\alpha \in PH^*(BGO_n, \mathbb{F}_2)$ can also be characterized as natural transformations from $|BGO_n|$ to $H^*(-, \mathbb{F}_2)$ on $QuasiProj_S$ such that $\alpha(T \otimes K) = \alpha(T)$ for any nondegenerate triple T and a line bundle K on a quasi-projective scheme X over S.

5 The main theorem

5.1 Statement of the main theorem

We are now able to state the main theorem, having made all the above preparation over étale cohomology and Gysin sequences for algebraic stacks. But the statement is just the algebraic stacky analogue of the corresponding topological theorem in [H-N-2], where the complex Lie groups $GO_n(\mathbb{C})$, etc. are replaced by the corresponding group schemes over S, and singular cohomology with coefficients $\mathbb{F}_2$ is replaced by étale cohomology with coefficients $\mathbb{F}_2$. We will adhere to the notation in [H-N-2] as much as possible.

Let $PH^*(BGO_n, \mathbb{F}_2) \subset H^*(BGO_n, \mathbb{F}_2)$ be the subring of primitive classes. Its elements are the universal cohomological invariants for nondegenerate quadric bundles of rank n. Let $B(v)^* : H^*(BGO_n) \to H^*(BO_{n-1})$ be the ring homomorphism induced by the group homomorphism $v : O_{n-1} \to GO_n$ defined by $g \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix}$. Let $(E_{n-1}, L_{n-1}, b_{n-1})$ be the universal triple over BGO_{n-1}. As shown above, the complement $L_{n-1}^\times$ of the zero section of L_{n-1} is isomorphic to BO_{n-1}, and there is a Gysin boundary map $d_{n-1} : H^*(BO_{n-1}) \to H^{*-1}(BGO_{n-1})$. Finally, let $\delta : PH^*(BGO_n) \to H^{*-1}(BGO_{n-1})$ be the composite linear map

$$PH^*(BGO_n) \hookrightarrow H^*(BGO_n) \overset{B(v)^*}{\to} H^*(BO_{n-1}) \overset{d_{n-1}}{\to} H^{*-1}(BGO_{n-1})$$

With the above notations, we have the following, where all cohomologies are smooth étale cohomologies with coefficients $\mathbb{F}_2$.

Theorem 11 *(**Main Theorem**) Let S be a regular scheme on which 2 is invertible. Let (X, Y) be a regular pair of stacks over S. Let Q be a quadric bundle on X, nondegenerate on $X - Y$ of rank $n \geq 2$, which is minimally degenerate on $Y \subset X$. Let $\mathbb{T}^{Q_Y}$ be the associated rank $n-1$ nondegenerate quadratic triple on Y. Let $\alpha \in PH^*(BGO_n)$ be a universal quadric invariant, and let $\alpha(Q_{X-Y}) \in H^*(X - Y)$ be its value on Q_{X-Y}. Let $\delta : PH^*(BGO_n) \to H^{*-1}(BGO_{n-1})$ be the linear map defined above, and let $(\delta(\alpha))(\mathbb{T}^{Q_Y})$ be the value of the resulting GO_{n-1}-characteristic class $\delta(\alpha)$ on $\mathbb{T}^{Q_Y}$. Let $\nu_Y(\det(Q)) \in H^0(Y)$ be the vanishing multiplicity along Y of the discriminant $\det(Q) \in \Gamma(X, L^n \otimes \det(E)^{-2})$. Then under the Gysin boundary map $\partial : H^*(X - Y) \to H^{*-1}(Y)$, we have the equality*

$$\partial(\alpha(Q_{X-Y})) = \nu_Y(\det(Q)) \cdot (\delta(\alpha))(\mathbb{T}^{Q_Y})$$

The rest of this paper is devoted to proving the theorem.

5.2 The algebraic stack of quadratic triples

Let $n \geq 1$. For any S-scheme X, let $\mathcal{M}_n(X)$ be the groupoid whose objects are all quadratic triples $T = (E, L, b)$ on X with $\mathrm{rank}(E) = n$, and a morphism $T \to T'$, where $T = (E, L, b)$ and $T' = (E', L', b')$, is pair of $\mathcal{O}_X$-linear isomorphisms $E \to E'$ and $L \to L'$ which takes b to b'. If $f : X' \to X$ is a morphism of S-schemes and T is a quadratic triple over X, the pull-back triple $f^*(T) = (f^*E, f^*L, f^*b)$ is defined in the obvious manner. For each $f : X' \to X$ and $f'' : X'' \to X'$ over S, we can define an isomorphism $\phi_{f,f'} : (ff')^*(T) \to f'^*f^*(T)$, which makes $\mathcal{M}_n$ into a groupoid-valued pseudo functor on S-schemes. It is clear that the pull-back f^* preserves both the dimension (rank of E) and the point-wise ranks (ranks of b_x), and pulls back the discriminant, so properties such as nondegenerateness, minimal degenerateness, etc. are preserved under pullbacks. The **almost nondegenerate triples**, which will mean triples whose point-wise rank is $\geq n-1$, form a full subgroupoid $\mathcal{M}$ of $\mathcal{M}_n$. The S-groupoid $\mathcal{M}$ has a full subgroupoid $\mathcal{D}$ formed by all the quadratic triples $T = (E, L, b)$ on S-schemes X such that $\det(T) = 0 \in \Gamma(X, L^n \otimes \det(E)^{-2})$ and $\mathrm{rank}(b_x) = n-1$ for all $x \in X$. This is the groupoid of minimally degenerate triples.

Proposition 12 *For any $n \geq 1$, almost nondegenerate triples of dimension n form an algebraic stack $\mathcal{M}$ over S, and minimally degenerate triples of dimension n form a closed substack $\mathcal{D} \subset \mathcal{M}$ defined by the vanishing of the discriminant of the universal triple $\mathcal{T}$ on $\mathcal{M}$. Moreover, $\mathcal{D}$ is a smooth relative divisor in $\mathcal{M}$ over S. The stack $\mathcal{M}$ is a global quotient stack under the reductive group scheme $GL_n \times \mathbb{G}_m$ over S.*

Proof We will see below that the S-groupoid $\mathcal{M}_n$ of all triples is an algebraic stack on S. It has a filtration by closed algebraic substacks $\mathcal{M}_n = \mathcal{M}_n^{\leq n} \supset \mathcal{M}_n^{\leq n-1} \supset \mathcal{M}_n^{\leq n-2} \supset \ldots$ where for any X, $\mathcal{M}_n^{\leq r}(X)$ is the groupoid of all triples (E, L, b) on X such that the determinants of minors of b of size $r+1$ vanish identically on X

(the relevant Fitting ideal is 0). The almost nondegenerate triples form the open substack

$$\mathcal{M} = \mathcal{M}_n - \mathcal{M}_n^{\leq n-2}.$$

The closed substack $\mathcal{D} \subset \mathcal{M}$ is given by

$$\mathcal{D} = \mathcal{M}_n^{\leq n-1} - \mathcal{M}_n^{\leq n-2}.$$

We now construct these stacks $\mathcal{M}_n^{\leq r}$. Let E be the universal vector bundle on the stack BGL_n and let L be the universal line bundle on $B\mathbb{G}_m$. On the stack $BGL_n \times B\mathbb{G}_m$ we get the vector bundles $p_1^*\underline{Sym}^2(E)$ and p_2^*L. Then $\mathcal{M}_n$ is the 'total space' (rather,'total stack') of the vector bundle corresponding to the locally free $\mathcal{O}$-module $\underline{Hom}(p_1^*\underline{Sym}^2(E), p_2^*L)$ on $BGL_n \times B\mathbb{G}_m$, which is an algebraic stack as it is a local construction in the sense of Laumon, Moret-Bailly [L-MB] Chapter 14. The closed substack $\mathcal{M}_n^{\leq r}(X)$ is defined by the vanishing of all determinants of minors of size $r+1$. In particular, $\mathcal{D}$ is defined in $\mathcal{M}$ by the vanishing of the discriminant, so it is a principal divisor.

As $BGL_n \times B\mathbb{G}_m$ is smooth over S, and as $\mathcal{M}_n$ is a geometric vector bundle over it, $\mathcal{M}_n$ is a smooth stack over S. The Jacobian criterion applied to the discriminant shows that $\mathcal{D}$ is a smooth relative divisor in $\mathcal{M}$ over S.

The affine space $\mathbf{A}_S^{n(n+1)/2}$ over S of $n \times n$-symmetric matrices has an action of $GL_n \times \mathbb{G}_m$ given in terms of valued points by $b \cdot (g, \lambda) = {}^t gbg\lambda^{-1}$. If $M \subset \mathbf{A}_S^{n(n+1)/2}$ is the open subscheme where $\mathrm{rank}(b) \geq n-1$, and if $D \subset M$ is the divisor defined by $\det(b) = 0$, then it can be seen that $\mathcal{M}$ is isomorphic to the quotient stack $[M/GL_n \times \mathbb{G}_m]$ under which $\mathcal{D}$ becomes its the closed substack $[D/GL_n \times \mathbb{G}_m]$. $\square$

Note that we have a tautological quadratic triple $\mathcal{T}_n = (\mathcal{E}_n, \mathrm{Ł}_n, \beta_n)$ on $(\mathcal{M}, \mathcal{D})$. If X is an algebraic stack, then to any almost nondegenerate quadratic triple T on X there functorially corresponds its **classifying morphism** $\chi_T : X \to \mathcal{M}$, together with an isomorphism of triples $\chi_T^*(\mathcal{T}_n) \to T$. The degeneration multiplicity along Y of a mildly degenerating triple T on a regular pair (X, Y) of stacks has the following obvious interpretation in terms of its classifying morphism $\chi_T : X \to \mathcal{M}$.

Proposition 13 *Let $T = (E, L, b)$ be a mildly degenerating triple on a regular pair (X, Y), with degeneration multiplicity $\nu_Y(T) : |Y| \to \mathbb{Z}_{\geq 0}$. Then under the classifying morphism $\chi_T : X \to \mathcal{M}$, the pull back of the closed substack $\mathcal{D} \subset \mathcal{M}$ is the closed substack $\nu_Y(T)Y \subset X$ defined by the ideal sheaf $(I_Y)^{\nu_Y(T)} \subset \mathcal{O}_X$.* $\square$

We now translate the main theorem in terms of the Gysin homomorphism $\partial : H^*(\mathcal{M} - \mathcal{D}) \to H^{*-1}(\mathcal{D})$. For this, we first define a 1-morphism $\rho : \mathcal{D} \to BGO_{n-1}$ as follows. If Y is any (affine) scheme over S, an object of the groupoid $\mathcal{D}(Y)$ is a minimally degenerate triple $T = (F, L, q)$ over Y, with $\mathrm{rank}(F) = n$ and $rank(q) = n-1$. Note that $\ker(q) \subset F$ is a line subbundle, and q induces a nondegenerate L-valued form $\overline{q}$ on the quotient $F/\ker(q)$. We get a new nondegenerate triple of rank $n-1$

$$\rho(F, L, q) = (F/\ker(q), L, \overline{q}) \otimes \ker(q)^{-1}$$

over Y. This is functorial in the triples and respects their pull-backs, so it defines 1-morphism of stacks $\rho : \mathcal{D} \to BGO_{n-1}$. Note that if $Q = [T]$ is a mildly degenerating quadric bundle on (X,Y), then

$$\mathbb{T}^{Q_Y} = \rho(T|_Y).$$

Hence the 1-morphism $\rho : \mathcal{D} \to BGO_{n-1}$ is the classifying morphism of the triple $\mathbb{T}^{\mathcal{Q}_\mathcal{D}}$ on $\mathcal{D}$, where $\mathcal{Q}$ is the mildly degenerating quadric bundle $\mathcal{Q} = [\mathcal{T}]$ on $(\mathcal{M},\mathcal{D})$ corresponding to the universal triple $\mathcal{T}$ on $\mathcal{M}$.

We will prove the main theorem in the following equivalent re-formulation.

Theorem 14 Main theorem restated in terms of the pair $(\mathcal{M},\mathcal{D})$: *The Gysin boundary map $\partial : H^*(\mathcal{M} - \mathcal{D}) \to H^{*-1}(\mathcal{D})$ on the primitive cohomology $PH^*(\mathcal{M} - \mathcal{D}) \subset H^*(\mathcal{M} - \mathcal{D})$ is given by the formula*

$$\partial|_{PH^*(\mathcal{M}-\mathcal{D})} = \rho^* \circ d_{n-1} \circ B(v)^*$$

where $B(v)^* : H^*(\mathcal{M} - \mathcal{D}) = H^*(BGO_n) \to H^*(BO_{n-1})$ is induced by $v : O_{n-1} \to GO_n : g \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix}$, the map $d_{n-1} : H^*(BO_{n-1}) \to H^{*-1}(BGO_{n-1})$ is the Gysin boundary map for the pair (BO_{n-1}, BGO_{n-1}), and $\rho : \mathcal{D} \to BGO_{n-1}$ is the 1-morphism defined by $\rho(F,L,q) = (F/\ker(q), L, \overline{q}) \otimes \ker(q)^{-1}$.

Proof of equivalence: Consider the universal quadratic triple $\mathcal{T}$ on $\mathcal{M}$, and the resulting quadric bundle $\mathcal{Q} = [\mathcal{T}]$, which is a mildly degenerating quadric bundle on the regular pair of stacks $(\mathcal{M},\mathcal{D})$, with degeneration multiplicity $\nu_\mathcal{D}(\det(\mathcal{Q})) = 1$. Let $\alpha \in PH^*(\mathcal{M} - \mathcal{D}) = PH^*(BGO_n)$. By definition, $\alpha(\mathcal{Q}_{\mathcal{M}-\mathcal{D}}) = \alpha$. Therefore the formula $\partial(\alpha) = \nu_Y(\det(Q)) \cdot (\delta(\alpha))(\mathbb{T}^{Q_Y})$ of the Theorem 11 applied to α and $\mathcal{Q}$ becomes $\partial(\alpha(\mathcal{Q}_{\mathcal{M}-\mathcal{D}})) = \delta(\alpha)(\mathbb{T}^{\mathcal{Q}_\mathcal{D}})$. As remarked above, $\mathbb{T}^{\mathcal{Q}_\mathcal{D}} = \rho(\mathcal{T}|_\mathcal{D})$. Hence we get for all $\alpha \in PH^*(\mathcal{M} - \mathcal{D})$ the equality $\partial(\alpha) = \rho^*\delta(\alpha) = \rho^* \circ d_{n-1} \circ B(v)^*(\alpha)$ by definition of the map δ from Theorem 11. As this holds for all $\alpha \in PH^*(\mathcal{M} - \mathcal{D})$, we get $\partial|_{PH^*(\mathcal{M}-\mathcal{D})} = \rho^* \circ d_{n-1} \circ B(v)^*$ as desired. (The above is a typical 'Yoneda argument'.)

Conversely, suppose the above formula holds. Given any mildly degenerating quadric Q on a regular pair of stacks (X,Y), let $\chi : X \to \mathcal{M}$ be the classifying 1-morphism for any chosen quadratic triple T on X which represents Q. Let $m = \nu_Y(Q)$. By Lemma 5.(2) we get a commutative diagram

$$\begin{array}{ccc}
H^i(\mathcal{M} - \mathcal{D}) & \stackrel{\partial_{(\mathcal{M},\mathcal{D})}}{\to} & H^{i-1}(\mathcal{D}) \\
{\scriptstyle \chi|^*_{X-Y}} \downarrow & & \downarrow {\scriptstyle m\cdot\chi|^*_Y} \\
H^i(X - Y) & \stackrel{\partial_{(X,Y)}}{\to} & H^{i-1}(Y)
\end{array}$$

We now make the following observation (15) before continuing with the proof that (11) implies (14).

15 Note that for any $\alpha \in H^i(\mathcal{M} - \mathcal{D})$, by definition $\alpha(T) = \chi|^*_{X-Y}(\alpha)$, and in particular when $\alpha \in PH^i(\mathcal{M} - \mathcal{D})$, by definition $\alpha(Q) = \chi|^*_{X-Y}(\alpha)$. Also, for any $\beta \in H^{i-1}(\mathcal{D})$, by definition $\beta(T|_Y) = \chi|^*_Y(\beta)$.

For any $\alpha \in PH^*(\mathcal{M}-\mathcal{D})$ we therefore have the following sequence of equalities.

$$\begin{aligned}
\partial_{(X,Y)}(\alpha(Q)) &= \partial_{(X,Y)}\chi|^*_{X-Y}(\alpha) \text{ by the Statement 15,}\\
&= m\cdot\chi|^*_Y\partial_{(\mathcal{M},\mathcal{D})}(\alpha) \text{ by the above application of Lemma 5.(2),}\\
&= m\cdot\chi|^*_Y\partial|_{PH^*(\mathcal{M}-\mathcal{D})}(\alpha) \text{ as by assumption } \alpha\in PH^*(\mathcal{M}-\mathcal{D}),\\
&= m\cdot\chi|^*_Y\rho^*\circ d_{n-1}\circ B(v)^*(\alpha) \text{ if Theorem 14 holds,}\\
&= m\cdot\chi|^*_Y\rho^*\delta(\alpha),\\
&\quad\text{where } \delta : PH^*(\mathcal{M}-\mathcal{D}) = PH^*(BGO_n) \to H^{*-1}(BGO_{n-1})\\
&\quad\text{is defined as earlier by } \delta = d_{n-1}\circ B(v)^*.
\end{aligned}$$

This gives the formula

$$\partial_{(X,Y)}(\alpha(Q)) = m\cdot\chi|^*_Y\rho^*\delta(\alpha)$$

Note that the composite 1-morphism $Y \xrightarrow{\chi|_Y} \mathcal{D} \xrightarrow{\rho} BGO_{n-1}$ is exactly the characteristic morphism $\chi' : Y \to BGO_{n-1}$ of the reduced triple $\mathbb{T}^{Q_Y} = (E/\ker(b), L, \overline{b}) \otimes \ker(b)^{-1}$ where $T = (E, L, b)$ represents Q. Hence we get

$$\chi|^*_Y\rho^*\delta(\alpha) = (\chi')^*\delta(\alpha) = \delta(\alpha)(\mathbb{T}^{Q_Y})$$

Putting together the previous two displayed equalities, we get

$$\partial_{(X,Y)}(\alpha(Q)) = \delta(\alpha)(\mathbb{T}^{Q_Y})$$

as desired. This completes the proof of the equivalence of Theorems 11 and 14. $\square$

5.3 Standard models for degenerating quadrics

Let K' be the universal line bundle on $B\mathbb{G}_m$. Consider any $n \geq 1$ and the universal triple $(E_{n-1}, L_{n-1}, b_{n-1})$ on the stack BGO_{n-1}. Consider the projections $p_1 : BGO_{n-1}\times B\mathbb{G}_m \to BGO_{n-1}$ and $p_2 : BGO_{n-1}\times B\mathbb{G}_m \to B\mathbb{G}_m$. Let $(\mathbf{E}, \mathbf{L}, \mathbf{b}) = p_1^*(E_{n-1}, L_{n-1}, b_{n-1})$ and $K = p_2^*K'$ be the pullbacks. Let the stack $\mathcal{N}$ be the total space of the line bundle $\mathbf{L}$ on the stack $\mathcal{Z} = BGO_{n-1}\times B\mathbb{G}_m$, and let $\pi : \mathcal{N} \to \mathcal{Z}$ be the projection 1-morphism. We regard $\mathcal{Z}$ as a closed substack $\mathcal{Z} \hookrightarrow \mathcal{N}$, embedded by the zero section. Note that the stacks $\mathcal{Z}$ and $\mathcal{N}$ are smooth over S.

We now define a certain quadratic triple on $\mathcal{N}$ following [H-N-2], called the **model triple**. Let $\tau \in \Gamma(\mathcal{N}, \pi^*(\mathbf{L}))$ be the tautological section, which vanishes exactly on the zero section $BGO_{n-1} \times B\mathbb{G}_m = \mathcal{Z} \subset \mathcal{N}$, with vanishing multiplicity 1. Then we get a quadratic triple (where $\oplus$ denotes the orthogonal direct sum)

$$T = ((\mathcal{O}_{\mathcal{N}}, \pi^*\mathbf{L}, \tau) \oplus \pi^*(\mathbf{E}, \mathbf{L}, \mathbf{b})) \otimes K$$

on $\mathcal{N}$ of dimension n, which has rank n on $\mathcal{N}-\mathcal{Z}$ and has rank $n-1$ on $\mathcal{Z}$, which means it is a mildly degenerating triple on $(\mathcal{N},\mathcal{Z})$, which is nondegenerate on $\mathcal{N}-\mathcal{Z}$ and minimally degenerate on the divisor $\mathcal{Z}$, with degeneration multiplicity $\nu_{\mathcal{Z}}(\det(T))=1$.

Let $\varphi:\mathcal{Z}\to\mathcal{D}$ be the classifying morphism of the degenerate triple

$$T_0=((\mathcal{O}_{\mathcal{Z}},\mathbf{L},0)\oplus(\mathbf{E},\mathbf{L},\mathbf{b}))\otimes K$$

which is the restriction of T to $\mathcal{Z}\hookrightarrow\mathcal{N}$. In the opposite direction, let the 1-morphism $\psi:\mathcal{D}\to\mathcal{Z}$ be defined as follows. For any S-scheme Y, an object of $\mathcal{D}(Y)$ is a minimally degenerate quadratic triple (F,L,q) on Y. Hence $\ker(q)$ is a line bundle on Y so it is an object of $B\mathbb{G}_m(Y)$, while $(F/\ker(q),L,\overline{q})$ (where $\overline{q}$ is induced by q) is an object of $BGO_{n-1}(Y)$. As $\mathcal{Z}=BGO_{n-1}\times B\mathbb{G}_m$, we can define

$$\psi(F,L,q)=((F/\ker(q),L,\overline{q})\otimes\ker(q)^{-1},\,\ker(q))\in Ob\,\mathcal{Z}(Y).$$

16 (Factorization of ρ) The 1-morphism $\rho:\mathcal{D}\to BGO_{n-1}$, defined earlier by $\rho(F,L,q)=(F/\ker(q),L,\overline{q})\otimes\ker(q)^{-1}$, factors as

$$\rho=p_1\circ\psi$$

where $p_1:BGO_{n-1}\times B\mathbb{G}_m\to BGO_{n-1}$ is the projection.

The composite 1-morphism $\varphi\circ\psi:\mathcal{D}\to\mathcal{D}$ is of importance to us. In terms of the above notation, we have

$$\varphi\circ\psi(F,L,q)=(\ker(q),L,0)\oplus(F/\ker(q),L,\overline{q}).$$

Lemma 17 *The 1-morphisms $\varphi:\mathcal{Z}\to\mathcal{D}$ and $\psi:\mathcal{D}\to\mathcal{Z}$ satisfy the following.*

(1) $\psi\circ\varphi=\mathrm{id}_{\mathcal{Z}}$.

(2) $\varphi\circ\psi:\mathcal{D}\to\mathcal{D}$ is $\mathbf{A}^1$-homotopic to $\mathrm{id}_{\mathcal{D}}$, that is, there exists a 1-morphism $F:\mathcal{D}\times\mathbf{A}^1\to\mathcal{D}$ such that $\varphi\circ\psi=F_0$ and $\mathrm{id}_{\mathcal{D}}=F_1$, where F_t denotes $F|_{\mathcal{D}\times\{t\}}$ for $t\in\mathbf{A}^1(\mathbb{Z}[1/2])=\mathbb{Z}[1/2]$. Hence by Remark 7, the cohomology map $\psi^*\circ\varphi^*:H^*(\mathcal{D})\to H^*(\mathcal{D})$ is identity.

(3) Consequently, the cohomology maps $\psi^*:H^*(\mathcal{Z})\to H^*(\mathcal{D})$ and $\varphi^*:H^*(\mathcal{D})\to H^*(\mathcal{Z})$ are isomorphisms, which are inverse to each other.

Proof Let $0\to E'\to E\to E''\to 0$ is a short exact sequence of vector bundles on an S-scheme Y. Let $\mathbf{A}^1$ be the affine line over with coordinate t. On $Y\times\mathbf{A}^1$ consider the homomorphism $g:p_Y^*E'\to p_Y^*E'\oplus p_Y^*E$ which sends $v\mapsto(tv,v)$. Then $\mathcal{F}=\mathrm{coker}(g)$ fits in a short exact sequence $0\to p_Y^*E'\to\mathcal{F}\to p_Y^*E''\to 0$ of vector bundles on $Y\times\mathbf{A}^1$, which specializes to the given exact sequence on Y for $t=1$ and specializes to the split exact sequence $0\to E'\to E'\oplus E''\to E''\to 0$ for $t=0$. The above construction is functorial, hence it also works for short exact sequences of vector bundles over algebraic stacks.

Applying this to the short exact sequence $0 \to \ker(b_n|_{\mathcal{D}}) \to \mathcal{E}_n|_{\mathcal{D}} \to \overline{\mathcal{E}_n|_{\mathcal{D}}} \to 0$ on $\mathcal{D}$ which arises from the universal minimally degenerate triple $\mathcal{T}_{\mathcal{D}} = (\mathcal{E}_n, \mathbb{L}_n, \beta_n)|_{\mathcal{D}}$, we get a short exact sequence $0 \to p_{\mathcal{D}}^* \ker(b_n|_{\mathcal{D}}) \to \mathcal{F} \to p_{\mathcal{D}}^*\overline{\mathcal{E}_n|_{\mathcal{D}}} \to 0$ of vector bundles on $\mathcal{D}\times \mathbf{A}^1$. Let $\widehat{b} : \mathcal{F}\otimes\mathcal{F} \to p_{\mathcal{D}}^* L$ be induced by $p_{\mathcal{D}}^*\overline{b}$. This defines a minimally degenerate triple $\widehat{\mathcal{T}_{\mathcal{D}}} = (\mathcal{F}, p_{\mathcal{D}}^* L, \widehat{b})$ on $\mathcal{D} \times \mathbf{A}^1$. Take $F = \chi_{\widehat{\mathcal{T}_{\mathcal{D}}}} : \mathcal{D} \times \mathbf{A}^1 \to \mathcal{D}$ to be its classifying morphism. $\square$

Let $\chi : \mathcal{N} \to \mathcal{M}$ be the classifying morphism of the mildly degenerating quadratic triple $T = ((\mathcal{O}_{\mathcal{N}}, \pi^*\mathbf{L}, \tau) \oplus \pi^*(\mathbf{E}, \mathbf{L}, \mathbf{b})) \otimes K$ on $(\mathcal{N}, \mathcal{Z})$ defined above. Under it, the pullback of $\mathcal{D} \subset \mathcal{M}$ is the zero section $\mathcal{Z} \subset \mathcal{N}$, with multiplicity 1. Both the pairs $(\mathcal{M}, \mathcal{D})$ and $(\mathcal{N}, \mathcal{Z})$ admit long exact Gysin sequences, and we have a commutative diagram as follows, where the vertical maps are induced by the classifying morphism of T and its restrictions.

$$\begin{array}{ccccccccc}
\ldots \to & H^i(\mathcal{M}) & \to & H^i(\mathcal{M}-\mathcal{D}) & \stackrel{\partial}{\to} & H^{i-1}(\mathcal{D}) & \to & H^{i+1}(\mathcal{M}) & \to \ldots \\
& \downarrow \chi^* & & \downarrow \chi|_{\mathcal{N}-\mathcal{Z}}^* & & \downarrow \chi|_{\mathcal{Z}}^* & & \downarrow \chi^* & \\
\ldots \to & H^i(\mathcal{N}) & \to & H^i(\mathcal{N}-\mathcal{Z}) & \stackrel{d}{\to} & H^{i-1}(\mathcal{Z}) & \to & H^{i+1}(\mathcal{N}) & \to \ldots
\end{array}$$

Note that $\mathcal{M}-\mathcal{D}$ is isomorphic to BGO_n, and $\mathcal{N}-\mathcal{Z}$ is isomorphic to $BO_{n-1}\times B\mathbb{G}_m$. Under these isomorphisms, the above map $\chi|_{\mathcal{N}-\mathcal{Z}}^* : H^i(\mathcal{M}-\mathcal{D}) \to H^i(\mathcal{N}-\mathcal{Z})$ is the cohomology map $H^i(BGO_n) \to H^i(BO_{n-1} \times B\mathbb{G}_m)$ associated to the 1-morphism $BO_{n-1} \times B\mathbb{G}_m \to BGO_n$ of stacks which is induced by the group scheme homomorphism

$$V : O_{n-1} \times \mathbb{G}_m \to GO_n : (g, \lambda) \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix} \lambda.$$

Hence the commutative diagram

$$\begin{array}{ccc}
H^i(\mathcal{M}-\mathcal{D}) & \stackrel{\partial_{(\mathcal{M},\mathcal{D})}}{\to} & H^{i-1}(\mathcal{D}) \\
\downarrow \chi|_{\mathcal{N}-\mathcal{Z}}^* & & \downarrow \chi|_{\mathcal{Z}}^* \\
H^i(\mathcal{N}-\mathcal{Z}) & \stackrel{\partial_{(\mathcal{N},\mathcal{Z})}}{\to} & H^{i-1}(\mathcal{Z})
\end{array}$$

becomes the commutative diagram

$$\begin{array}{ccc}
H^i(BGO_n) & \stackrel{\partial_{(\mathcal{M},\mathcal{D})}}{\to} & H^{i-1}(\mathcal{D}) \\
B(V)^* \downarrow & & \downarrow \varphi^* \\
H^i(BO_{n-1} \times B\mathbb{G}_m) & \stackrel{d}{\to} & H^{i-1}(BGO_{n-1} \times B\mathbb{G}_m)
\end{array}$$

18 (Factorization of the Gysin map $\partial : H^i(\mathcal{M}-\mathcal{D}) \to H^{i-1}(\mathcal{D})$) As φ^* and ψ^* are inverses by Lemma 17, the above commutative diagram gives the following commutative diagram.

$$\begin{array}{ccc}
H^i(BGO_n) & \stackrel{\partial_{(\mathcal{M},\mathcal{D})}}{\to} & H^{i-1}(\mathcal{D}) \\
B(V)^* \downarrow & & \uparrow \psi^* \\
H^i(BO_{n-1} \times B\mathbb{G}_m) & \stackrel{d}{\to} & H^{i-1}(BGO_{n-1} \times B\mathbb{G}_m)
\end{array}$$

Lemma 19 *Let $PH^*(BGO_n) \subset H^*(BGO_n)$ be the primitive subring. Then we have a commutative diagram*

$$\begin{array}{ccc}
PH^i(BGO_n) & \hookrightarrow & H^i(BGO_n) \\
\downarrow {\scriptstyle B(v)^*} & & \downarrow {\scriptstyle B(V)^*} \\
H^i(BO_{n-1}) & \stackrel{{p'_1}^*}{\to} & H^i(BO_{n-1} \times B\mathbb{G}_m)
\end{array}$$

where ${p'_1}^*$ is induced by the projection $p'_1 : BO_{n-1} \times B\mathbb{G}_m \to BO_{n-1}$.

Proof Let (F, q) be the universal quadratic form on BO_{n-1}, where F is a vector bundle of rank $n-1$ and q is a nondegenerate quadratic form on F with values in $\mathcal{O}_{BO_{n-1}}$. Let K be the universal line bundle on $B\mathbb{G}_m$. The morphism $B(V) : BO_{n-1} \times B\mathbb{G}_m \to BGO_n$ is the classifying map of the nondegenerate quadratic triple $T = ((\mathcal{O}, \mathcal{O}, 1) \oplus (F, \mathcal{O}, q)) \otimes K$. If $\alpha \in H^*(BGO_n)$ then $B(V)^*(\alpha) = \alpha(T)$. As $\alpha \in PH^*(BGO_n)$, we have $\alpha(T) = \alpha(T \otimes K^{-1}) = \alpha((\mathcal{O}, \mathcal{O}, 1) \oplus (F, \mathcal{O}, q)) = B(v)^*(\alpha)$. □

5.4 Completion of the proof of the main theorem

20 Let $p_1 : BGO_{n-1} \times B\mathbb{G}_m \to BGO_{n-1}$ and ${p'}_1 : BO_{n-1} \times B\mathbb{G}_m \to BO_{n-1}$ be the projections. Then the following diagram commutes, where the horizontal maps are Gysin boundary maps.

$$\begin{array}{ccc}
H^i(BO_{n-1}) & \stackrel{d_{n-1}}{\to} & H^{i-1}(BGO_{n-1}) \\
{\scriptstyle {p'}_1^*} \downarrow & & \downarrow {\scriptstyle p_1^*} \\
H^i(BO_{n-1} \times B\mathbb{G}_m) & \stackrel{d}{\to} & H^{i-1}(BGO_{n-1} \times B\mathbb{G}_m)
\end{array}$$

As before, we identify $\mathcal{M} - \mathcal{D}$ with BGO_n, under which $PH^*(\mathcal{M} - \mathcal{D})$ gets identified with $PH^*(BGO_n) \subset H^*(BGO_n)$. We then have the following sequence of equalities.

$$\begin{aligned}
\partial|_{PH^*(BGO_n)} &= \psi^* \circ d \circ B(V)^* \text{ by Statement 18,} \\
&= \psi^* \circ d \circ {p'}_1^* \circ B(v)^* \text{ by Lemma 19,} \\
&= \psi^* \circ p_1^* \circ d_{n-1} \circ B(v)^* \text{ by Statement 20,} \\
&= \rho^* \circ d_{n-1} \circ B(v)^* \text{ by Statement 16.}
\end{aligned}$$

Hence the Gysin boundary map $\partial_{(\mathcal{M},\mathcal{D})} : H^*(\mathcal{M} - \mathcal{D}) \to H^{*-1}(\mathcal{D})$ satisfies the formula

$$\partial|_{PH^*(\mathcal{M}-\mathcal{D})} = \rho^* \circ d_{n-1} \circ B(v)^*$$

which completes the proof of Theorem 14, and hence of Theorem 11. □

5.5 The case of a separably closed base field $k = k_{sep}$

When our base S is of the form Spec k for a separably closed field k of characteristic $\neq 2$, the rings $H^*(BGO_{n,k}, \mathbb{F}_2)$ and $PH^*(BGO_{n,k}, \mathbb{F}_2)$ are the same as in the topological case by [Bh], so the Theorem 11 now implies that all the calculations in [H-N-2] apply without change over such a k.

In [Bh], it was shown that the key topological facts on which the the calculation of $H^*_{sing}(BGO_{2n}(\mathbb{C}), \mathbb{F}_2)$ in [H-N-1] is based have suitable analogs in the algebraic category over a separably closed base field k of characteristic $\neq 2$, and hence we get the following determination of $H^*(BGO_{n,k}, \mathbb{F}_2)$.

First, recall that the cohomology ring $H^*(BGL_{n,k}, \mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[\overline{c}_1, \ldots, \overline{c}_n]$ in n variables $\overline{c}_i \in H^{2i}(BGL_{n,k}, \mathbb{F}_2)$ which are the **universal mod 2 Chern classes**. Also, the cohomology ring $H^*(BO_{n,k}, \mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[w_1, \ldots, w_n]$ in n variables $w_i \in H^i(BGL_{n,k}, \mathbb{F}_2)$ which are the **universal Stiefel-Whitney classes** of Laborde [La] over $k = k_{sep}$ (see, for example, [E-K-V]). Under the homomorphism $H^*(BGL_{n,k}, \mathbb{F}_2) \to H^*(BO_{n,k}, \mathbb{F}_2)$ induced by the inclusion $O_{n,k} \hookrightarrow GL_{n,k}$, we have $\overline{c}_i \mapsto w_i^2$.

21 The rings $H^*(BGO_{2n+1,k}, \mathbb{F}_2)$. For the odd special orthogonal group scheme $SO_{2n+1,k}$ we have $H^*(BSO_{2n+1,k}, \mathbb{F}_2) = \mathbb{F}_2[w_2, \ldots, w_{2n+1}]$, as follows from the direct product decomposition $O_{2n+1} = \mu_2 \times SO_{2n+1,k}$ by using the Künneth formula (see section 5 of [Bh] for details of the argument). We have an isomorphism $\mathbb{G}_m \times SO_{2n+1} \to GO_{2n+1}$ defined in terms of valued points by $(\lambda, g) \mapsto \lambda g$. It follows that $BGO_{2n+1} = B\mathbb{G}_m \times BSO_{2n+1}$. Hence we have

$$H^*(BGO_{2n+1,k}, \mathbb{F}_2) = \mathbb{F}_2[c, w_2, \ldots, w_{2n+1}]$$

where c denotes the pullback of the first Chern class $\overline{c}_1$ from $H^2(BG_{m,k}, \mathbb{F}_2)$, and the $w_2, \ldots, w_{2n+1}$ are the pull-backs of the Hasse-Witt classes in $H^*(BSO_{2n+1}, \mathbb{F}_2)$. Note that $c, w_2, \ldots, w_{2n+1}$ are algebraically independent over $\mathbb{F}_2$, and are homogeneous of degrees $2, 2, 3, \ldots, 2n+1$.

22 The rings $H^*(BGO_{2n,k}, \mathbb{F}_2)$. This ring is generated by the cohomology classes λ, a_{2i-1}, b_{4j}, and d_T over $\mathbb{F}_2$, where $i, j \in \{1, \ldots, n\}$ and $T \subset \{1, \ldots, n\}$ has cardinality $|T| \geq 2$. These are defined as follows. Let (E_{2n}, L_{2n}, b_{2n}) denote the universal quadratic triple on BGO_{2n}. Then $\lambda = \overline{c}_1(L_{2n})$ and $b_{4j} = \overline{c}_{2j}(E_{2n})$. Next, consider the Gysin boundary map $d : H^*(BO_{n,k}, \mathbb{F}_2) \to H^{*-1}(BGO_{n,k}, \mathbb{F}_2)$ that we introduced earlier. Then $a_{2i-1} = d(w_{2i}) \in H^{2i-1}(BGO_{n,k}, \mathbb{F}_2)$, and for any $T = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$, we have $d_T = d(w_{2i_1} \cdots w_{2i_r}) \in H^{2(i_1+\ldots+i_r)-1}(BGO_{n,k}, \mathbb{F}_2)$. These generators satisfy relations given explicitly in [H-N-1] Theorem 3.9.

23 Relation with Stiefel-Whitney and Chern classes. When $k = k_{sep}$, the ring homomorphisms $H^*(BGL_{n,k}, \mathbb{F}_2) \to H^*(BGO_{n,k}, \mathbb{F}_2) \to H^*(BO_{n,k}, \mathbb{F}_2)$,

induced by the inclusions $O_n \hookrightarrow GO_n \hookrightarrow GL_n$, are given by the same explicit formulas in terms of the generators of $H^*(BGO_{n,k}, \mathbb{F}_2)$ which are deduced in Section 3 of [H-N-2] in the topological category for $H^*_{sing}(BGO(\mathbb{C}), \mathbb{F}_2)$.

24 With the above explicit descriptions of $H^*(BGO_{n,k}, \mathbb{F}_2)$ in the even and odd cases, all the calculations Section 8 of [H-N-2] hold in the algebraic category over such a k. These include Corollaries 8.1 and 8.2 which give formulas in terms of generators and relations for the 'even rank degenerating to odd rank' and 'odd rank degenerating to even rank' cases of the Theorem 11, illustrated with complete lists of generators of the primitive rings $PH^*(BGO_{n,k}, \mathbb{F}_2)$ and their images under ∂ in all ranks up to $n = 6$.

Remark 25 For any scheme X over $\mathbb{Z}[1/2]$ and any $y \in \Gamma(X, \mathcal{O}_X)^\times$, let $(y) \in H^1(X, \mathbb{F}_2)$ denote the Kummer class of y. If X lies over a separably closed field base field of characteristic $\neq 2$, then note that $(y)^2 = 0 \in H^2(X, \mathbb{F}_2)$, and moreover, $(-1) = 0 \in H^2(X, \mathbb{F}_2)$. These properties of the Kummer class also hold in the topological category for Kummer classes of nowhere vanishing complex valued continuous functions y. These properties, which are crucially used in [H-N-1] and [H-N-2], are not available over an arbitrary base such as $\mathbb{Z}[1/2]$. In fact all powers κ^n of the Kummer class $\kappa = (-1) \in H^1(\mathbb{Z}[1/2], \mathbb{F}_2)$ are nonzero, as may be seen by base change to $\mathbb{R}$. This is one of the reasons why the explicit determination of the rings $H^*(BGO_n, \mathbb{F}_2)$ and $PH^*(BGO_n, \mathbb{F}_2)$ in terms of generators and relations is more complicated over a general base. This question will be addressed elsewhere.

References

[Be] Behrend, K.A. : Derived ℓ-adic categories for algebraic stacks. Mem. Amer. Math. Soc. 163 (2003), no. 774

[Bh] Bhaumik, S. : Characteristic classes for $GO(2n)$ in étale cohomology. Preprint, 2012. To appear in Proc. Indian Acad. Sci. (http://arxiv.org/abs/1201.4628)

[EGA-IV$_2$] Grothendieck, A. : Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, seconde partie. Inst. Hautes Études Sci. Publ. Math. No. 24, 1964.

[E-K-V] Esnault, H., Kahn, B., Viehweg, E. : Coverings with odd ramification and Stiefel-Whitney classes. J. Reine Angew. Math. 441 (1993), 145-188.

[Fu] Fujiwara, K. : A proof of the absolute purity conjecture (after Gabber). Algebraic geometry 2000, Azumino (Hotaka), 153-183, Adv. Stud. Pure Math., 36, Math. Soc. Japan, 2002.

[Ju] Jouanolou, J. P. : Une suite exacte de Mayer-Vietoris en K-théorie algébrique. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, 1972), 293-316. Lecture Notes in Math., 341, Springer, 1973.

[H-N-1] Holla, Y.I. and Nitsure, N. : Characteristic classes for $GO(2n, \mathbb{C})$, Asian Journal

of Mathematics 5, 2001, 169-182.

[H-N-2] Holla, Y.I. and Nitsure, N. : Topology of quadric bundles. Internat. J. Math. 12 (2001), 1005-1047.

[La] Laborde, O. : Classes de Stiefel-Whitney en cohomologie tale. Colloque sur les Formes Quadratiques (Montpellier, 1975). Bull. Soc. Math. France Suppl. Mem. No. 48 (1976), 4751.

[L-MB] Laumon, G. and Moret-Bailly, L. : *Champs algbriques.* Springer 2000.

[L-O] Laszlo, Y. and Olsson, M. : The six operations for sheaves on Artin stacks I: Finite coefficients. Publ. Math. IHÉS 107 (2008), 109-168.

[Na-Ra] Narasimhan, M.S. and Ramanan, S. : Geometry of Hecke cycles -I. *C. P. Ramanujam - a tribute*, 291-345, TIFR Studies in Mathematics 8, Springer, 1978.

[N-1] Nitsure, N. : Topology of conic bundles. J. London Math. Soc. (2) 35 (1987) 18-28.

[N-2] Nitsure, N. : Cohomology of desingularization of moduli space of vector bundles. Compositio Math. 69 (1989) 303-339.

[O] Olsson, M. : Sheaves on Artin stacks. J. Reine Angew. Math. 603 (2007), 55-112.

[Se] Seshadri, C.S. : Geometric reductivity over arbitrary base. Advances in Math. 26 (1977), 225-274.

[SGA 4] Artin, M., Grothendieck, A., Verdier, J.-L. (ed.) : *Théorie des topos et cohomologie étale des schḿas*, Sem. Geom. Alg. Bois Marie 1963-64, LNM 269, 270 and 305, Springer 1972-73.

[SGA $4\frac{1}{2}$] Deligne, P. : *Cohomologie étale.* (With the collaboration of J. F. Boutot, A. Grothendieck, L. Illusie and J.-L. Verdier.) LNM 569, Springer 1977.

[Td] Toda, H. : Cohomology of Classifying Spaces. In the volume *Homotopy Theory and Related Topics* (Editor H. Toda), Advances in Pure Math. **9** (1986) 75-108.

[Tot] Totaro, B. : Chow ring of a classifying space, *Algebraic K-theory* (Seattle 1997) 249-281, Proc. Symp. in Pure Math. 67, Amer. Math. Soc., 1999.

Saurav Bhaumik <saurav@math.tifr.res.in> and Nitin Nitsure <nitsure@math.tifr.res.in>

Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India.

01 February 2013, revised on 24 April 2013